Quantum IF-THEN is commuting or classical

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October 7, 2025

The `if-then` construct in declarative programming is a transformation that takes a Boolean condition ϕ and some operation P, and returns the operation that consists of first determining whether ϕ is true, and if so applying P, and otherwise doing nothing. Quantum mechanically, Boolean propositions correspond to subspaces Π of a system's Hilbert space, and operations correspond to unitary operators U, and we may ask what the quantum analogue of the `if-then` construct is. Any operation $\mathcal{E}$ worthy of being called a conditional of the form `if` $\Pi : U$ ought to satisfy

$$\mathcal{E} : |\psi\rangle \mapsto U|\psi\rangle \text{ if } |\psi\rangle \in \Pi \tag{1}$$

$$\mathcal{E} : |\psi\rangle \mapsto |\psi\rangle \text{ if } |\psi\rangle \in \Pi^{\perp} \tag{2}$$

It is clear that we may simply perform the projective measurement that distinguishes a subspace Π from its orthogonal complement, and then perform a unitary operation U in the event that the (classical) measurement outcome is `True`. But we know that there are more fully quantum options as well. For example, if the Boolean condition is that a control qubit be in the state $|1\rangle$, and U acts on a disjoint set of qubits, then the familiar controlled-U operation satisfies the axioms of a conditional operation. More generally, if Π is invariant under the action of U, the unitary operation $U\Pi + I - \Pi$ (abusing notation so that Π denotes both a subspace and the corresponding projector) satisfies the conditions above, as does the non-unitary channel with Kraus operators $U\Pi$ and $I - \Pi$, which is just the classical measurement and feedback channel. One might wonder whether there is anything "more quantum" than this classical measurement and feedback channel in the event that Π is not invariant under U. The answer turns out to be no.

Claim 1. *Let $\Pi \subseteq \mathcal{H}$ be a subspace and U a unitary operator. Suppose that $\mathcal{E}$ is a quantum* `if` *$\Pi : U$ channel. If Π is not invariant under U, then $\mathcal{E}$ must be the channel that performs a projective measurement of the subspace Π and then applies U if the outcome is* `True` *and does nothing if the outcome is* `False`*.*

Proof. As it is a quantum channel, $\mathcal{E}$ may be implemented by first applying an isometry $V : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}_{\text{aux}}$ and then tracing out the auxiliary system. For any $|\psi\rangle \in \Pi$, we must have

$$V|\psi\rangle = U|\psi\rangle \otimes |\chi(\psi)\rangle, \tag{3}$$

and in fact the auxiliary state $|\chi(\psi)\rangle$ cannot depend on ψ at all, as otherwise V would generate entanglement with the auxiliary system. Thus there is some $|\chi_\Pi\rangle$ such that for any $|\psi\rangle \in \Pi$,

$$V|\psi\rangle = U|\psi\rangle \otimes |\chi_\Pi\rangle. \tag{4}$$

similarly, there is some $|\chi_{\Pi^\perp}\rangle$ such that for any $|\phi\rangle \in \Pi^\perp$ we have

$$V|\phi\rangle = |\phi\rangle \otimes |\chi_{\Pi^\perp}\rangle. \tag{5}$$

Suppose that U does not preserve the subspace Π. Then there are states $|\psi\rangle \in \Pi$ and $|\phi\rangle \in \Pi^\perp$ such that

$$\langle\phi|U|\psi\rangle \neq 0. \tag{6}$$

Because V is an isometry, we require

$$\langle\phi|V^\dagger V|\psi\rangle = 0. \tag{7}$$

We may also write

$$\langle\phi|V^\dagger V|\psi\rangle = \left(\langle\phi| \otimes \langle\chi_{\Pi^\perp}|\right)\left(U|\psi\rangle \otimes |\chi_\Pi\rangle\right) = \langle\phi|U|\psi\rangle \langle\chi_{\Pi^\perp}|\chi_\Pi\rangle . \tag{8}$$

Therefore, we have

$$\langle\chi_{\Pi^\perp}|\chi_\Pi\rangle = 0. \tag{9}$$

Thus $\mathcal{E}$ is equivalent to measuring Π and applying U or identity conditioned on the classical outcome. □

An interpretation of this is that whereas when Π and U commute, a valid conditional exists that does not result in information leakage to the environment, when they do not, the final state of the auxiliary system increases in entropy by exactly the classical entropy of the initial system state with respect to the measurement Π.