Bell's Inequality

Jeffrey Epstein

March 11, 2018

These notes follow Bell's original papers on the subject.

Theorem 1. *A local hidden variable theory (a hidden variable theory in which the setting of one measurement apparatus does not affect the outcome of another) must obey the inequality*

$$|(ab)_\sigma - (bc)_\sigma| \leq 1 + (ac)_\sigma. \tag{1}$$

where $(ab)_\sigma = \langle A(a)B(b)\rangle_\sigma$ *is the correlation between two* ± 1*-valued random variables* $A_\sigma(a)$ *and* $B_\sigma(b)$ *that are perfectly anticorrelated when* $a = b$.

Proof. Suppose we have a theory for describing a particular kind of thing. In our theory, these things are described by some object we'll call a preparation. I'm going to give you a bunch of things that are "the same" in the sense that they are all described by the same preparation π. You can perform an operation on these things that results in two ± 1-valued random variables $A_\pi(a)$ and $B_\pi(b)$, where a and b are settings you are free to choose. If you wanted, you could calculate the correlation between these variables:

$$\langle A(a)B(b)\rangle_\pi . \tag{2}$$

We'd like to design a deterministic hidden-variable theory to reproduce the results of our theory. By this we mean that there is a "hidden variable" λ such that the values of the random variables are determined by λ. The preparations π determine probability distributions $\rho_\pi(\lambda)$, so that we may write the correlations as follows:

$$\langle A(a)B(b)\rangle_\pi = \int \rho_\pi(\lambda) A(a,\lambda) B(b,\lambda) d\lambda. \tag{3}$$

Suppose that the things I give you are described by a preparation σ such that the random variables $A_\sigma(a)$ and $B_\sigma(b)$ are perfectly anti-correlated when $a = b$:

$$\langle A(a)B(a)\rangle_\sigma = -1 \longleftrightarrow A_\sigma(a) = -B_\sigma(a). \tag{4}$$

Then for arbitrary settings a, b, and c, we can do some calculations:

$$\langle A(b)B(a)\rangle_\sigma - \langle A(b)B(c)\rangle_\sigma = \int \rho_\sigma(\lambda) \left[A(b,\lambda)B(a,\lambda) - A(b,\lambda)B(c,\lambda)\right] d\lambda \tag{5}$$

$$= \int \rho_\sigma(\lambda) \left[A(b,\lambda)A(c,\lambda) - A(b,\lambda)A(a,\lambda)\right] d\lambda \tag{6}$$

$$= \int \rho_\sigma(\lambda) A(b,\lambda)A(a,\lambda) \left[A(a,\lambda)A(c,\lambda) - 1\right] d\lambda. \tag{7}$$

Taking the absolute value and using the fact that the random variables take values ± 1, we have

$$|\langle A(b)B(a)\rangle_\sigma - \langle A(b)B(c)\rangle_\sigma| \leq \int \rho_\sigma(\lambda) \left[1 - A(a,\lambda)A(c,\lambda)\right] d\lambda \tag{8}$$

$$= 1 + \int \rho_\sigma(\lambda) A(a,\lambda) B(c,\lambda) d\lambda \tag{9}$$

$$= 1 + \langle A(a)B(c)\rangle_\sigma . \tag{10}$$

Noticing that $\langle A(a)B(b)\rangle_\pi = \langle A(b)B(a)\rangle_\pi$, we denote this quantity $(ab)_\pi$. Then the inequality reads

$$|(ab)_\sigma - (bc)_\sigma| \leq 1 + (ac)_\sigma. \tag{11}$$

□

Theorem 2. *The predictions of quantum mechanics cannot be explained by a local hidden variable theory.*

Proof. Consider a system of two spins. Let the preparation σ be the singlet state

$$|\sigma\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle\right). \tag{12}$$

Let the random variables $A_\pi(a)$ and $B_\pi(b)$ be the outcomes of measurements of $(\sigma \cdot a) \otimes \mathbb{1}$ and $\mathbb{1} \otimes (\sigma \cdot b)$, respectively, for unit vectors a and b. Then the correlations are

$$(ab)_\sigma = \langle A(a)B(b)\rangle_\sigma = \langle\sigma|\,(\sigma \cdot a) \otimes (\sigma \cdot b)\,|\sigma\rangle = -a \cdot b. \tag{13}$$

Then for any a, $(aa)_\sigma = -1$, as in the statement of Bell's theorem. If there is to be an explanatory local hidden variable theory, then Bell's inequality must always hold:

$$|-a \cdot b + b \cdot c| \leq 1 - a \cdot c. \tag{14}$$

Consider the following measurements:

$$a = (1,0,1)/\sqrt{2} \qquad b = (0,0,1) \qquad c = (1,0,0). \tag{15}$$

Then the inequality reads

$$\left|-1/\sqrt{2}\right| \leq 1 - 1/\sqrt{2} \longrightarrow 1 \leq \sqrt{2} - 1 \longrightarrow 2 \leq \sqrt{2}. \tag{16}$$

This is a contradiction. □