This document derives four conserved quantities of the electromagnetic field in vacuum, the energy, momentum, angular momentum, and a fourth related to boosts. The procedure followed is to reformulate Maxwell's equations in terms of a variational principle (specified by the Maxwell Lagrangian) and to derive the Noether charges corresponding to spacetime translations and Lorentz transformations.

The electromagnetic field in three dimensional space and in the absence of matter is governed by the vacuum Maxwell's equations. In Heaviside-Lorentz units and setting the speed of light $c = 1$, these are

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0 \tag{1}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{2}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{3}$$

$$\boldsymbol{\nabla} \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 0. \tag{4}$$

This document will demonstrate that the following are conserved quantities for solutions of the Maxwell equations:

$$H = \int \left(\mathbf{E}^2 + \mathbf{B}^2\right) d^3\mathbf{x} \tag{5}$$

$$\mathbf{P} = \int \mathbf{E} \times \mathbf{B} \, d^3\mathbf{x} \tag{6}$$

$$\mathbf{L} = \int \mathbf{x} \times (\mathbf{E} \times \mathbf{B}) \, d^3\mathbf{x} \tag{7}$$

$$\mathbf{Q} = \int \left(\frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) \mathbf{x} - t\,(\mathbf{E} \times \mathbf{B})\right) d^3\mathbf{x}. \tag{8}$$

Potential formulation

Any divergence-free vector field may be written as the curl of another vector field, and any curl-free vector field may be written as the gradient of a scalar field, so if $\mathbf{E}$ and $\mathbf{B}$ solve Maxwell's equations, we may find some vector field $\mathbf{A}$ and some scalar field V such that

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{9}$$

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}. \tag{10}$$

If we take $\mathbf{A}$ and V to be the basic dynamical variables of the theory, with electric and magnetic fields defined in terms of these as above, then the second and third Maxwell equations are automatically satisfied, and the first and fourth take the form

$$0 = \boldsymbol{\nabla} \cdot \left(-\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla} V\right) = -\frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \mathbf{A} - \boldsymbol{\Delta} V = \Box V - \frac{\partial}{\partial t}\left(\frac{\partial V}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{A}\right) \tag{11}$$

$$0 = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) - \frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla} V\right) = \Box \mathbf{A} + \boldsymbol{\nabla}\left(\frac{\partial V}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{A}\right) \tag{12}$$

where $\Box = \partial_t^2 - \boldsymbol{\Delta}$ is the d'Alembertian operator. If we define the 4-vector potential $A^\mu = (V, \mathbf{A})$ and the differential operator $\partial_\mu = (\partial_t, \boldsymbol{\nabla})$, with indices raised and lowered by the metric $\eta = \mathrm{diag}(+1, -1, -, 1, -1)$, we find that these two equations take the compact form

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = 0. \tag{13}$$

Thus given any 4-vector field A^μ that satisfies the equation of motion (13), the electric and magnetic fields derived from it via (9) and (10) will satisfy Maxwell's equations. Moreover, any solution of Maxwell's equations corresponds to such an A^μ. (In fact, a solution of Maxwell's equations corresponds to a family of

4-potentials A^μ. This so-called gauge-invariance turns out to be very important, but won't be used in this derivation.) It will be useful to define the field-strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0. \end{pmatrix} \tag{14}$$

Note that it must be shown or postulated separately of any of the analysis performed so far that the electric and magnetic fields actually transform as a rank-2 tensor under Lorentz transformations (although it is indeed the case that they do).

Variational formulation

Consider a physical system described, within some spatial region $\mathcal{R}$ and time interval $t_1 \leq t \leq t_2$, by a 4-vector field A^μ that vanishes at the spatial boundary $\partial\mathcal{R}$. One way to pick out the physically possible trajectories of the system is to declare that they are those trajectories that are stationary points of some functional with respect to variations that vanish at both spatial and temporal boundaries. We take this functional to be the action

$$S = \int_{t_1}^{t_2} dt \int_{\mathcal{R}} d^3\mathbf{x}\, \mathcal{L} \tag{15}$$

defined in terms of the Lagrangian

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right). \tag{16}$$

The variation of the action is

$$\delta S = \int_{t_1}^{t_2} dt \int_{\mathcal{R}} d^3\mathbf{x}\, \delta\mathcal{L} = \int_{t_1}^{t_2} dt \int_{\mathcal{R}} d^3\mathbf{x}\, \frac{\partial \mathcal{L}}{\partial(\partial^\tau A^\omega)} \delta(\partial^\tau A^\omega) = 2\int_{t_1}^{t_2} dt \int_{\mathcal{R}} d^3\mathbf{x} \left(\partial_\tau A_\omega - \partial_\omega A_\tau\right) \partial^\tau (\delta A^\omega) \tag{17}$$

$$= -2\int_{t_1}^{t_2} dt \int_{\mathcal{R}} d^3\mathbf{x} \left[\partial^\tau \left(\partial_\tau A_\omega - \partial_\omega A_\tau\right)\right] \delta A^\omega \tag{18}$$

so that the stationarity condition is equivalent to the equation of motion

$$\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = 0. \tag{19}$$

Noether's Theorem

We can examine the properties of stationary trajectories A^μ, and we will find that the following conservation laws hold:

$$0 = \partial_\mu \mathcal{J}^{\mu\nu} = \partial_\mu \mathcal{M}^{\mu\lambda\tau} \tag{20}$$

where the conjugate momentum $\Pi^{\mu\nu}$, stress tensor $\mathcal{J}^{\mu\nu}$, and angular momentum $\mathcal{M}^{\mu\lambda\tau}$ are defined as follows:

$$\Pi^{\mu\nu} = -(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{21}$$

$$\mathcal{J}^{\mu\nu} = \Pi^{\mu\lambda}\partial^\nu A_\lambda - \eta^{\mu\nu}\mathcal{L} \tag{22}$$

$$\mathcal{M}^{\mu\lambda\tau} = x^\tau \mathcal{J}^{\mu\lambda} - x^\lambda \mathcal{J}^{\mu\tau} + \Pi^{\mu\lambda}A^\tau - \Pi^{\mu\tau}A^\lambda. \tag{23}$$

The vanishing of the 4-divergences provides conservation laws as follows. We may consider the $\mu = 0$ component integrated over space, and we find

$$\frac{d}{dt}\int \mathcal{J}^{0\nu}\, d\mathbf{x} = \int \partial_0 \mathcal{J}^{0\nu}\, d\mathbf{x} = -\int \partial_i \mathcal{J}^{i\nu}\, d\mathbf{x} = 0, \tag{24}$$

where $i = 1, 2, 3$ ranges over the spatial coordinates and we have used the assumption that the fields vanish at infinity. Then we have the conserved quantities

$$Q^\nu = \int \mathcal{J}^{0\nu}\, d\mathbf{x}. \tag{25}$$

Similar reasoning may be applied to the angular momentum $\mathcal{M}^{\mu\lambda\tau}$ to obtain conserved quantities $Q^{\lambda\tau}$.

Stress tensor from translational invariance We'll work first towards finding the stress tensor by exploiting the invariance of the Lagrangian under translations, i.e. its lack of explicit dependence on x^μ. Imagine that an observer has a notebook in which they have written down the value of the Lagrangian at each point in spacetime. Denote by $\mathcal{L}$ this function of x^μ. Now suppose the same observer shifts the origin of their coordinate system by an infinitessimal shift ϵ^μ, so that a point that was assigned the coordinates x^μ in the old system is now assigned the coordinates $x^\mu - \epsilon^\mu$. In this new coordinate system, the observer records the values of the Lagrangian for all values of the (new) coordinate. Refer to this function of x^μ as $\mathcal{L}_{\text{neq}}$. We can now use two methods to evaluate

$$\delta\mathcal{L} = \mathcal{L}_{\text{new}} - \mathcal{L}. \tag{26}$$

First, we can just view the Lagrangian as a scalar field in its own right, so that

$$\mathcal{L}_{\text{new}}(x^\mu) = \mathcal{L}(x^\mu - \epsilon^\mu) = -\epsilon^\mu \partial_\mu \mathcal{L}. \tag{27}$$

On the other hand, we $\mathcal{L}$ depends on the derivatives $\partial_\mu A^\nu$, so that

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \delta(\partial_\mu A_\nu) = \Pi^{\mu\nu}\delta(\partial_\mu A_\nu) = \partial_\mu\left(\Pi^{\mu\nu}\delta A_\nu\right) = -\partial_\mu\left(\Pi^{\mu\nu}\epsilon^\lambda \partial_\lambda A_\nu\right), \tag{28}$$

where we have used the definition of the conjugate momentum, the fact that $\delta(\partial_\mu A_\nu) = \partial_\mu(\delta A_\nu)$, and the previously-established divergenceless of the conjugate momentum, i.e. the equation of motion (13). Equating these two expressions we find

$$0 = \partial_\mu\left(\Pi^{\mu\nu}\epsilon^\lambda \partial_\lambda A_\nu\right) - \epsilon^\mu \partial_\mu \mathcal{L} = \epsilon_\lambda \partial_\mu\left(\Pi^{\mu\nu}\partial^\lambda A_\nu - \eta^{\mu\lambda}\mathcal{L}\right), \tag{29}$$

which must hold for any ϵ^λ, establishing

$$\partial_\mu\left(\Pi^{\mu\nu}\partial^\lambda A_\nu - \eta^{\mu\lambda}\mathcal{L}\right) = 0 \tag{30}$$

as claimed.

Angular momentum tensor from Lorentz invariance In order to obtain the angular momentum tensor, we imagine instead that the observer undergoes a Lorentz transformation. Now we have

$$\mathcal{L}_{\text{new}}(x^\mu) = \mathcal{L}(x^\mu - \epsilon^\mu_\nu x^\nu) = \mathcal{L}(x^\mu) - \epsilon^\lambda_\nu x^\nu \partial_\lambda \mathcal{L}(x^\mu) \tag{31}$$

so that we have

$$\delta\mathcal{L} = \mathcal{L}_{\text{new}}(x^\mu) - \mathcal{L}(x^\mu) = -\epsilon^\lambda{}_\nu\, x^\nu\, \partial_\lambda \mathcal{L}(x) = -\partial^\lambda\left(\epsilon_{\lambda\nu} x^\nu \mathcal{L}\right) \tag{32}$$

As before, we also have

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)}\delta A_\nu\right) = \partial_\mu\left(\Pi^{\mu\nu}\delta A_\nu\right). \tag{33}$$

This time we have

$$A_{\nu\ \text{new}}(x^\lambda) = \left(\delta^\tau_\nu - \epsilon^\tau_\nu\right) A_\tau\left(x^\lambda - \epsilon_{\lambda\sigma}x^\sigma\right) = \left(\delta^\tau_\nu - \epsilon^\tau_\nu\right)\left(A_\tau(x) - \epsilon_{\lambda\sigma}x^\sigma \partial^\lambda A_\tau(x)\right) = A_\nu - \epsilon_{\lambda\sigma}x^\sigma\partial^\lambda A_\nu - \epsilon^\tau_\nu A_\tau \tag{34}$$

so that

$$\delta A_\nu = -\epsilon_{\lambda\sigma} x^\sigma \partial^\lambda A_\nu - \epsilon_\nu^\tau A_\tau = -\epsilon_\lambda^\tau x_\tau \partial^\lambda A_\nu - \epsilon_\lambda^\tau \delta_\nu^\lambda A_\tau = -\epsilon^{\lambda\tau} \left(x_\tau \partial_\lambda A_\nu + \eta_{\lambda\nu} A_\tau \right) \tag{35}$$

Plugging this in to the variation of the Lagrangian:

$$\delta \mathcal{L} = \partial_\mu \left(\Pi^{\mu\nu} \delta A_\nu \right) = -\partial_\mu \left(\Pi^{\mu\nu} \epsilon^{\lambda\tau} \left(x_\tau \partial_\lambda A_\nu + \eta_{\lambda\nu} A_\tau \right) \right) \tag{36}$$

$$= -\epsilon_{\lambda\tau} \partial_\mu \left(\Pi^{\mu\nu} \left(x^\tau \partial^\lambda A_\nu + \delta_\nu^\lambda A^\tau \right) \right) = -\epsilon_{\lambda\tau} \partial_\mu \left(x^\tau \mathcal{J}^{\mu\lambda} + \eta^{\mu\lambda} x^\tau \mathcal{L} + \Pi^{\mu\nu} \delta_\nu^\lambda A^\tau \right) \tag{37}$$

equating the two expressions:

$$0 = -\epsilon_{\lambda\tau} \partial_\mu \left(x^\tau \mathcal{J}^{\mu\lambda} + \eta^{\mu\lambda} x^\tau \mathcal{L} + \Pi^{\mu\nu} \delta_\nu^\lambda A^\tau \right) + \partial^\lambda \left(\epsilon_{\lambda\nu} x^\nu \mathcal{L} \right) = -\epsilon_{\lambda\tau} \partial_\mu \left(x^\tau \mathcal{J}^{\mu\lambda} + \Pi^{\mu\nu} \delta_\nu^\lambda A^\tau \right). \tag{38}$$

As $\epsilon^{\lambda\tau}$ is an arbitrary antisymmetric tensor, we have

$$0 = \partial_\mu \left(x^\tau \mathcal{J}^{\mu\lambda} - x^\lambda \mathcal{J}^{\mu\tau} + \Pi^{\mu\lambda} A^\tau - \Pi^{\mu\tau} A^\lambda \right) \tag{39}$$

as claimed.

Translating back to E and B

Now we may convert back into the language of electric and magnetic fields from the language of the 4-potential. The Lagrangian density for electromagnetism is

$$\mathcal{L} = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} \left(\mathbf{E}^2 - \mathbf{B}^2 \right) \tag{40}$$

so that the conjugate momentum density is

$$\Pi^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -(\partial^\mu A^\nu - \partial^\nu A^\mu) = -F^{\mu\nu}. \tag{41}$$

The energy-momentum tensor is

$$\mathcal{J}^{\mu\nu} = \Pi^\mu{}_\lambda \partial^\nu A^\lambda - g^{\mu\nu} \mathcal{L} = -g^{\mu\alpha} (\partial_\alpha A_\lambda - \partial_\lambda A_\alpha) \partial^\nu A^\lambda - g^{\mu\nu} \mathcal{L} = -g^{\mu\alpha} F_{\alpha\lambda} \partial^\nu A^\lambda - g^{\mu\nu} \mathcal{L} \tag{42}$$

so that we have for the time-time component

$$\mathcal{J}^{00} = -g^{0\alpha} F_{\alpha\lambda} \partial^0 A^\lambda - \mathcal{L} = -F_{0\lambda} \partial^0 A^\lambda - \frac{1}{2} \left(\mathbf{E}^2 - \mathbf{B}^2 \right) = -\mathbf{E} \cdot \dot{\mathbf{A}} - \frac{1}{2} \left(\mathbf{E}^2 - \mathbf{B}^2 \right) \tag{43}$$

$$= -\mathbf{E} \cdot (-\mathbf{E} - \boldsymbol{\nabla} V) - \frac{1}{2} \left(\mathbf{E}^2 - \mathbf{B}^2 \right) = \frac{1}{2} \left(\mathbf{E}^2 + \mathbf{B}^2 \right) + \mathbf{E} \cdot \boldsymbol{\nabla} V \tag{44}$$

$$= \frac{1}{2} \left(\mathbf{E}^2 + \mathbf{B}^2 \right) + \boldsymbol{\nabla} \cdot (V \mathbf{E}) \tag{45}$$

where we've used the divergenceless nature of the electric field. The time-space component becomes

$$\mathcal{J}^{0i} = -g^{0\alpha} F_{\alpha\lambda} \partial^i A^\lambda = -F_{0\lambda} \partial^i A^\lambda = -\mathbf{E} \cdot \partial_i \mathbf{A} = -E_j \partial_i A_j = (\mathbf{E} \times \mathbf{B})_i + \boldsymbol{\nabla} \cdot (A_i \mathbf{E}) \tag{46}$$

where the last equality uses the identity for the product of two Levi-Civita symbols and the divergenceless nature of the electric field.

The angular momentum tensor is

$$\mathcal{M}^{\mu\lambda\tau} = x^\tau \mathcal{J}^{\mu\lambda} - x^\lambda \mathcal{J}^{\mu\tau} + \Pi^{\mu\lambda} A^\tau - \Pi^{\mu\tau} A^\lambda \tag{47}$$

so that the time-space-space component is

$$\begin{aligned}
\mathcal{M}^{0ij} &= x^j \mathcal{J}^{0i} - x^i \mathcal{J}^{0j} + \Pi^{0i} A^j - \Pi^{0j} A^i && (48)\\
&= x^j \mathcal{J}^{0i} - x^i \mathcal{J}^{0j} - F^{0i} A^j + F^{0j} A^i && (49)\\
&= x^j \mathcal{J}^{0i} - x^i \mathcal{J}^{0j} + E^i A^j - E^j A^i && (50)\\
&= x^j \left(\epsilon_{ikl} E^k B^l + \nabla \cdot (A_i \mathbf{E})\right) - x^i \mathcal{J}^{0j} + E^i A^j - E^j A^i && (51)\\
&= \epsilon_{ikl} x^j E^k B^l + \nabla \cdot (A_i \mathbf{E}) x^j - x^i \mathcal{J}^{0j} + E^i A^j - E^j A^i && (52)\\
&= \epsilon_{ikl} x^j E^k B^l + \partial_k (A_i E_k x^j) - A_i E_j - x^i \mathcal{J}^{0j} + E^i A^j - E^j A^i && (53)\\
&= \epsilon_{ikl} x^j E^k B^l + \partial_k (A_i E_k x^j) - \epsilon_{jkl} x^i E^k B^l - \partial_k (A_j E_k x^i) && (54)\\
&= x^j (\mathbf{E} \times \mathbf{B})^i - x^i (\mathbf{E} \times \mathbf{B})^j + \boldsymbol{\nabla} \cdot \left(\mathbf{E}\left(A_i x^j - A_j x^i\right)\right) && (55)
\end{aligned}$$

The time-time-space component becomes

$$\begin{aligned}
\mathcal{M}^{00i} &= x^i \mathcal{J}^{00} - x^0 \mathcal{J}^{0i} + \Pi^{00} A^i - \Pi^{0i} A^0 && (56)\\
&= x^i \left(\frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) + \boldsymbol{\nabla} \cdot (V \mathbf{E})\right) - x^0 \left((\mathbf{E} \times \mathbf{B})_i + \boldsymbol{\nabla} \cdot (A_i \mathbf{E})\right) + \Pi^{00} A^i - \Pi^{0i} A^0 && (57)\\
&= x^i \left(\frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) + \boldsymbol{\nabla} \cdot (V \mathbf{E})\right) - x^0 \left((\mathbf{E} \times \mathbf{B})_i + \boldsymbol{\nabla} \cdot (A_i \mathbf{E})\right) + E^i A^0 && (58)\\
&= \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) x^i - t\,(\mathbf{E} \times \mathbf{B})_i + x^i \boldsymbol{\nabla} \cdot (V \mathbf{E}) - t \boldsymbol{\nabla} \cdot (A_i \mathbf{E}) + E^i A^0 && (59)\\
&= \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) x^i - t\,(\mathbf{E} \times \mathbf{B})_i + \partial_j \left(V x^i E_j\right) - V E_i - t \boldsymbol{\nabla} \cdot (A_i \mathbf{E}) + E^i A^0 && (60)\\
&= \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) x^i - t\,(\mathbf{E} \times \mathbf{B})_i + \partial_j \left(V x^i E_j\right) - \partial_j (t A_i E_j) && (61)\\
&= \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right) x^i - t\,(\mathbf{E} \times \mathbf{B})_i + \boldsymbol{\nabla} \cdot \left(\mathbf{E}\left(V x^i - t A_i\right)\right). && (62)
\end{aligned}$$

These quantities may now be integrated, discarding the total divergence terms, to obtain the conserved quantities claimed at the beginning of the document.