Representation Theorem for Rank-m Isotropic Tensors in Dimension n

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This note on isotropic tensors is Section 1 of the Appendix of [1]. If you find it useful, please cite that paper.

It is well known that the pair $(\boldsymbol{\delta}, \boldsymbol{\epsilon})$, where $\boldsymbol{\delta}$ is the rank two Kronecker tensor and $\boldsymbol{\epsilon}$ is the rank n Levi-Civita or fully antisymmetric tensor, generate all isotropic tensors on dimension n. Recall that the components of these tensors are

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j, \end{cases} \tag{1}$$

and

$$\epsilon_{i_1 \ldots i_n} = \begin{cases} 1 & i_1 \ldots i_n \text{ even permutation of } 1 \ldots \text{n} \\ -1 & i_1 \ldots i_n \text{ odd permutation of } 1 \ldots \text{n} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The sense in which these are generators is made clearer by the graphical calculus frequently used for tensor manipulation in the tensor network community, see for example [2] for an introduction to this streamlined language. Briefly, rank m tensors are represented by boxes with m "legs", each representing one of the indices. Connecting two legs corresponds to identifying the corresponding indices and performing a summation over all possible values of the index, so that for example the scalar product of two rank four tensors is represented pictorially by the diagram in Fig. 1a. The absence of "free legs" indicates that this is indeed a scalar. In this graphical language, the representation theorem is simple to state: a basis for the isotropic tensors of order m in n dimensions may be obtained by contracting m indexed legs with copies of the $\boldsymbol{\delta}$ and $\boldsymbol{\epsilon}$ tensors in all possible ways, which we may visualize as in Fig. 1b. For example, in two dimensions we can construct the isotropic rank four tensors drawn in Fig. 1c, which are written in component notation as $\delta_{ij}\epsilon_{kl}$, $\delta_{il}\delta_{kj}$, and $\delta_{ik}\delta_{jl}$. We note the curious circumstance, due to the fact that viscosity is rank four and the Levi-Civita tensor is rank n, that only in dimensions two and four is there a possibility of a component of the viscosity manifesting breaking of mirror symmetry.

Proofs of the fact that $\boldsymbol{\delta}$ and $\boldsymbol{\epsilon}$ tensors generate all isotropic tensors in dimension n appear in both [3] and [4], but the former reference proves a more general theorem using more powerful machinery, while the latter is (for the authors) somewhat difficult to follow. We provide here for convenience a compact proof. The essential idea is simply that the two geometrical quantities that are preserved by rotations of n-dimensional Euclidean space are the inner product between pairs of vectors and the signed volume of the parallelepiped spanned by n vectors. These correspond to the tensors $\boldsymbol{\delta}$ and $\boldsymbol{\epsilon}$. The only point that must be verified is that nothing else is preserved. We approach the proof in three steps.

Let $v_1, \ldots, v_m$ and $u_1, \ldots, u_m$ be ordered m-tuples of vectors in $\mathbb{R}^n$. Let $\langle w_1, w_2 \rangle$ denote the inner product of vectors w_1 and w_2 and $[w_1, w_2, \ldots, w_n]$ denote the determinant of the matrix whose i^{th} column is w_i. Then the following hold:

1. *If $\langle v_i, v_j \rangle = \langle u_i, u_j \rangle$ for all i, j, then for some $Q \in O(n)$ we have $u_i = Qv_i$ for all i.*

2. *If moreover $[v_{i_1}, v_{i_2}, \ldots, v_{i_n}] = [u_{i_1}, u_{i_2}, \ldots, u_{i_n}]$ for all n-tuples of index assignments, then for some $Q \in SO(n)$ we have $u_i = Qv_i$ for all i.*

We can establish this result by making use of the notion of Gram-Schmidt orthogonalization. Consider forming the $n \times n$ matrix E_0 as follows:

1. Begin by setting E_0 equal to the $n \times 1$ column vector v_1.

2. For i from 2 to m, check whether v_i is in the column space of E_0, and if it is not, append v_i as the right-most column of E_0.

3. If E_0 has fewer than n columns, append columns to the right such that the column space of E_0 is $\mathbb{R}^n$.

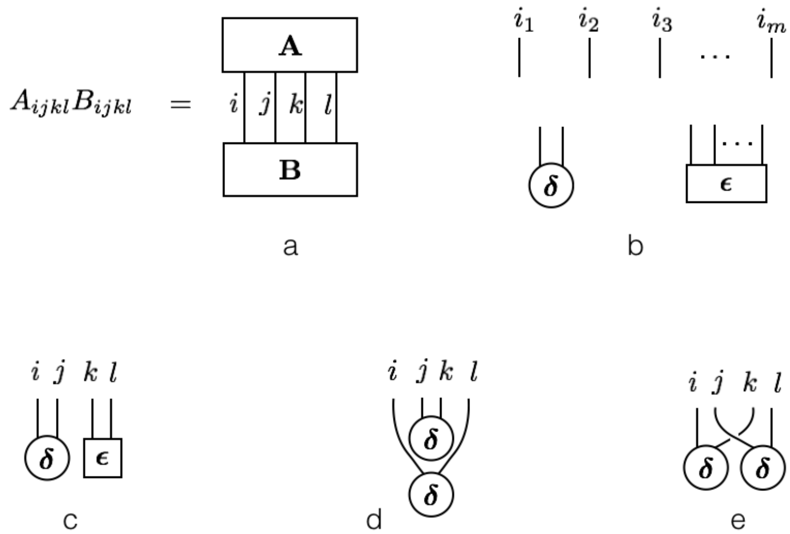

FIG. 1. In (a) the scalar product of two fourth-order tensors is represented graphically. In (b) the components for the construction of a basis for all m^{th}-order isotropic tensors in some dimension n are presented. The idea is that the indexed legs should be connected with copies of the two generator tensors $\boldsymbol{\delta}$ and $\boldsymbol{\epsilon}$. The set of all possible such diagrams spans the space of isotropic tensors. The tensor $\delta_{ij}\epsilon_{kl}$ is represented graphically in (c), $\delta_{il}\delta_{jk}$ in (d), and $\delta_{ik}\delta_{jl}$ in (e). Note that the "crossing" of the two legs in the third diagram is purely for visual convenience. There is no difference between "left over right" and "right over left".

Now we may define an orthogonal matrix $E \in O(n)$ by performing Gram-Schmidt orthogonalization on the columns of E_0. Let the columns of E be denoted by e_α, $\alpha = 1, 2, \ldots n$. The $n \times m$ matrix V with columns v_i may then be expressed as $V = EM$, where $M_{\alpha j} = \langle e_\alpha, v_j \rangle$.

We may proceed analogously with the u_i to define matrices $\tilde{E}$, $\tilde{M}$, and U such that $U = \tilde{E}\tilde{M}$. Because the inner products $\langle e_\alpha, v_j \rangle$ and $\langle \tilde{e}_\alpha, u_j \rangle$ depend only on pairwise inner products of the v_i and the u_i, respectively, we may conclude that $M = \tilde{M}$. Then $U = \tilde{E}M = \tilde{E}E^T EM = \tilde{E}E^T V \equiv QV$, so that $u_i = \tilde{E}E^T v_i$ for all i. Because E and $\tilde{E}$ (and their transposes) are orthogonal, so is $Q = \tilde{E}E^T$, establishing the first point above.

In order to establish the second point, we need to consider the determinant of $Q = \tilde{E}E^T$. Because orthogonal matrices have determinant ± 1 and the Gram-Schmidt procedure uses matrix column operations that do not change the sign of the determinant, we have

$$\det E = \operatorname{sgn} \det E_0 \tag{3}$$
$$\det \tilde{E} = \operatorname{sgn} \det \tilde{E}_0. \tag{4}$$

Then $\tilde{E}E^T \in SO(n)$ if and only if $\det E = \det \tilde{E}$.

If the v_i and u_i independently span $\mathbb{R}^n$, then all of the columns of E_0 and $\tilde{E}_0$ are members of these sets of vectors. Then the determinants are equal if and only if $[v_{i_1}, v_{i_2}, \ldots, v_{i_n}] = [u_{i_1}, u_{i_2}, \ldots, u_{i_n}]$ for all choices $i_1, \ldots, i_n$. If the v_i and u_i do not span $\mathbb{R}^n$, we are free to choose the extra basis vectors, so may arrange to have $\det(\tilde{E}E^T) = 1$. In this situation, $[v_{i_1}, v_{i_2}, \ldots, v_{i_n}] = [u_{i_1}, u_{i_2}, \ldots, u_{i_n}] = 0$. This establishes the second point above.

This fact allows us to prove the following:

Suppose that f is a function of m-tuples of vectors in $\mathbb{R}^n$ that is invariant under the orthogonal group $O(n)$ in the sense that $f(Qv_1, \ldots, Qv_n) = f(v_1, \ldots, v_n)$ for any $Q \in O(n)$. Then f can be expressed as a function of the pairwise inner products $\langle v_i, v_j \rangle$. If f is only required to be invariant under the special orthogonal group $SO(n)$, then f may be expressed as a function of the pairwise inner products $\langle v_i, v_j \rangle$ and the determinants $[v_{i_n}, \ldots, v_{i_n}]$.

Let f be invariant under $O(n)$. Suppose that $\langle v_i, v_j\rangle = \langle u_i, u_j\rangle$ for all i, j and that $f(u_1, \ldots, u_m) \neq f(v_1, \ldots v_m)$. By the result given above, the equality of pairwise inner products implies that there is some $Q \in O(n)$ such that $u_i = Qv_i$ for all i. Then $f(Qv_1, \ldots, Qv_m) \neq f(v_1, \ldots, v_m)$. But this contradicts the assumption that f is invariant under $O(n)$. Therefore, f must be completely determined by the set of pairwise inner products $\langle v_i, v_j\rangle$.

Now let f be invariant under $SO(n)$ but not necessarily under all of $O(n)$. Suppose that $\langle v_i, v_j\rangle = \langle u_i, u_j\rangle$ and $[v_{i_n}, \ldots, v_{i_n}] = [u_{i_n}, \ldots, u_{i_n}]$ for all choices of index assignments $f(u_1, \ldots, u_m) \neq f(v_1, \ldots v_m)$. By the above result, there is some $Q \in SO(n)$ such that $u_i = Qv_i$ for all i, so $f(Qv_1, \ldots, Qv_m) \neq f(v_1, \ldots, v_m)$. Again, this leads to a contradiction, so f must be determined by the set of inner products and determinants.

Now we are equipped to prove the promised representation theorem:

Theorem 1. *Let T be an order m tensor on dimension n, in other words a multilinear map*

$$T : \underbrace{\mathbb{R}^n \otimes \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n}_{m \text{ times}} \to \mathbb{R}. \tag{5}$$

Suppose that T is invariant under the special orthogonal group $SO(n)$ of proper rotations in the sense that for any $Q \in SO(n)$ and any $u_1, u_2, \ldots, u_m \in \mathbb{R}^n$, we have

$$T(Qu_1 \otimes Qu_2 \otimes \cdots \otimes Qu_m) = T(u_1 \otimes u_2 \otimes \cdots \otimes u_m). \tag{6}$$

Then T may be expressed as a linear combination of products of order 2 Kronecker tensors δ and at most one order n alternating/Levi-Civita tensors ϵ acting on disjoint sets of tensor factors (indices). If T is invariant under the entire orthogonal group $O(n)$, then it is expressible only in terms of Kronecker tensors.

By invariance under $SO(n)$, we can conclude using the previous result that $T(u_1 \otimes u_2 \otimes \cdots \otimes u_m)$ is determined completely the pairwise inner products $\langle u_i, u_j\rangle$ and volumes $[u_{i_1}, \ldots, u_{i_n}]$. For T to be linear in each of its arguments, this function must be a linear combination of products of inner products and volume forms in which each argument u_i appears exactly once. The inner product is given by $\langle u, v\rangle = \delta(u \otimes v)$ with δ the Kronecker tensor, and the volume form is given by $[u, v, \ldots, w] = \epsilon(u \otimes v \otimes \cdots \otimes w)$ with ϵ the alternating tensor. It is easy to see that the alternating tensor itself is odd under reflections so that a product of two alternating tensors on disjoint sets of indices is even. Then such products may be expressed solely in terms of Kronecker tensors. This establishes the first part of the theorem. If T is required to be invariant under $O(n)$, it can't contain any terms with an odd number of ϵ tensors, as these are odd under parity-inverting transformations. This establishes the second, and the representation theorem is proven.

This representation theorem provides a method for generating and studying the possible isotropic viscosity tensors, as we may use it to construct orthonormal bases for the full space of isotropic rank four tensors in two dimensions that diagonalize any symmetry of interest. In two dimensions, the space of isotropic rank four tensors is six-dimensional. Two different orthogonal bases (eigenbases for different sets of symmetries) for this space are presented in Tables I and II.

[1] J. M. Epstein and K. K. Mandadapu, (2019).
[2] J. C. Bridgeman and C. T. Chubb, Journal of physics A: Mathematical and theoretical **50**, 223001 (2017).
[3] H. Weyl, *The classical groups: their invariants and representations*, Vol. 1 (Princeton university press, 1946).
[4] H. Jeffreys, in *Mathematical Proceedings of the Cambridge philosophical society*, Vol. 73 (Cambridge University Press, 1973) pp. 173–176.

Basis Tensor	Components	$j \leftrightarrow l$	$ij \leftrightarrow kl$	P
$\mathbf{e}^{(1)}$	$\delta_{ij}\delta_{kl} - \epsilon_{ij}\epsilon_{kl}$	+	+	+
$\mathbf{e}^{(2)}$	$\epsilon_{ik}\epsilon_{jl}$	-	+	+
$\mathbf{e}^{(3)}$	$\delta_{ik}\delta_{jl}$	+	+	+
$\mathbf{e}^{(4)}$	$\epsilon_{ik}\delta_{jl}$	+	-	-
$\mathbf{e}^{(5)}$	$\epsilon_{ij}\delta_{kl} + \epsilon_{kl}\delta_{ij}$	+	+	-
$\mathbf{e}^{(6)}$	$\epsilon_{jl}\delta_{ik}$	-	-	-

TABLE I. Basis for the isotropic rank four tensors in two dimensions in which the index permutations $j \leftrightarrow l$ and $i \leftrightarrow k$, $j \leftrightarrow l$ are diagonal. Components of the viscosity odd under the former do not contribute to the momentum balance, while components odd under the latter contribute to the odd viscosity. The mirror transformation $(x_1, x_2) \mapsto (-x_1, x_2)$ is also diagonal in this basis. Note that the $+(-)$ indicates that the basis tensor is even (odd) under the indicated transformation. The basis tensors are orthogonal (but not normalized) with respect to the inner product $A_{ijkl}B_{ijkl}$.

Basis Tensor	Components	$i \leftrightarrow j$	$k \leftrightarrow l$	$ij \leftrightarrow kl$	P
$\mathbf{s}^{(1)}$	$\delta_{ij}\delta_{kl}$	+	+	+	+
$\mathbf{s}^{(2)}$	$\delta_{ik}\delta_{j\ell} - \epsilon_{ik}\epsilon_{jl}$	+	+	+	+
$\mathbf{s}^{(3)}$	$\epsilon_{ij}\epsilon_{kl}$	-	-	+	+
$\mathbf{s}^{(4)}$	$\epsilon_{ik}\delta_{j\ell} + \epsilon_{j\ell}\delta_{ik}$	+	+	-	-
$\mathbf{s}^{(5)}$	$\epsilon_{ik}\delta_{j\ell} - \epsilon_{j\ell}\delta_{ik} + \epsilon_{ij}\delta_{k\ell} + \epsilon_{k\ell}\delta_{ij}$	-	+	N/A	-
$\mathbf{s}^{(6)}$	$\epsilon_{ik}\delta_{j\ell} - \epsilon_{j\ell}\delta_{ik} - \epsilon_{ij}\delta_{k\ell} - \epsilon_{k\ell}\delta_{ij}$	+	-	N/A	-

TABLE II. Basis for the isotropic rank four tensors in two dimensions in which the index permutation $i \leftrightarrow j$, the permutation $k \leftrightarrow l$, and again the mirror transformation are diagonal. All but two of these basis elements are also eigenvectors of the permutation $i \leftrightarrow k$, $j \leftrightarrow l$, so where possible we also indicate the eigenvalue of the basis tensors under this symmetry. It is this basis we use for discussing the Green-Kubo relations presented in the main text.